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New exact solutions of a nonlinear cross-diffusion system

Roman Cherniha[1,2] **and Liliia Myroniuk**[2]

[1] Institute of Mathematics, NAS of Ukraine, 3 Tereshchenkivs'ka Street, 01601 Kyiv, Ukraine
[2] Faculty of Mathematics, Lesya Ukraïnka Volyn National University, 13 Voly Avenue, 43025 Lutsk, Ukraine

E-mail: cherniha@imath.kiev.ua and Liliia_Myroniuk@univer.lutsk.ua

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Abstract
A system of two reaction-diffusion equations with cross-diffusion and quadratic nonlinearities is considered. The system is a particular case, the well-known model proposed by Shigesada *et al* (1979 *J. Theor. Biol.* **79** 83). New non-Lie ansätze (special-type substitutions) reducing the system to systems of first-order ordinary differential equations are obtained and applied to find its exact solutions. Several families of exact solutions are constructed, their asymptotic behavior is investigated and some biological interpretation of the solutions is provided. Lie symmetry of this system is also discussed.

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1. Introduction

In 1979, Shigesada *et al* [1] proposed a mathematical model to describe the densities of two biological species, which takes into account the heterogeneity of the environment and nonlinear dispersive movements of the individuals of these populations. The model was developed on the basis of Morisita's phenomelogical theory of environmental density and has the form

$$\begin{aligned} u_t &= [(d_1 + d_{11}u + d_{12}v)u]_{xx} + (W_x u)_x + u(a_1 - b_1 u - c_1 v), \\ v_t &= [(d_2 + d_{21}u + d_{22}v)v]_{xx} + (W_x v)_x + v(a_2 - b_2 u - c_2 v), \end{aligned} \tag{1}$$

where the functions u and v arising in system (1) give the densities of two competing species in space and time, d_1 and d_2 denote the diffusion coefficients, $d_{12}v$ and $d_{21}u$ are so-called cross-diffusion pressures, $d_{11}u$ and $d_{22}v$ are intra-diffusion pressures, a_1 and a_2 are the intrinsic growth coefficients, b_1 and c_2 denote the coefficients of intra-specific competitions, b_2 and c_1 denote the coefficients of inter-specific competitions. The function $W(x)$ is so-called environmental potential, which is assumed to be known. Obviously, this system with $d_{ij} = 0, i = 1, 2, j = 1, 2$ and $W(x) = \text{const}$ reduces to the classical diffusive Lotka–Volterra

(LV) system. Nevertheless, the authors of [1] assumed that the environmental potential may be a non-constant function, system (1) with $W(x) = \text{const}$ is usually referred to as the Shigesada–Kawasaki–Teramoto (SKT) system/model.

It was shown by numerical simulations that system (1) possesses solutions describing coexistence of the species by the spatial segregation of habitat [1]. This kind of coexistence results from the mutual interferences of the species and the heterogeneity of the environment and means a steady-state segregation of densities of two competing species. The existence of the steady-state segregation clearly depends on the initial distributions of u and v and the parameter values of (1).

Starting from the pioneer works [2, 3], the conditions of existence, uniqueness and global stability/instability of solutions for the diffusive LV system and SKT system were investigated by many authors (see [4–7] and the papers cited therein). Nevertheless there are only a few papers where the relevant exact solutions are constructed in explicit form [8–11]. Note that a wide range of exact solutions of LV-type systems with power diffusivities (without cross-diffusion) were constructed in the recently published papers [12, 13].

In this paper system (1) with $W(x) = \text{const}$, i.e. the SKT system

$$\begin{aligned} u_t &= [(d_1 + d_{11}u + d_{12}v)u]_{xx} + u(a_1 - b_1u - c_1v), \\ v_t &= [(d_2 + d_{21}u + d_{22}v)v]_{xx} + v(a_2 - b_2u - c_2v) \end{aligned} \tag{2}$$

is considered, which is usually studied instead of (1) [4–6]. Depending on the signs of the parameters a_k, b_k and c_k $(k = 1, 2)$ the SKT system (2) can describe different types of species interactions (competition, mutualism, prey–predator interaction). Hereafter $d_{ij}, i = 1, 2, j = 1, 2$ are assumed to be real constants and $d_{12}^2 + d_{21}^2 \neq 0$, i.e., we consider only the systems with cross-diffusion.

This paper is devoted to constructing new exact solutions of SKT system (2) and to investigation of their asymptotic behavior. The paper is organized as follows. In section 2, we use the method of additional generating conditions [14, 15] for constructing non-Lie ansätze that reduce system (2) to the systems of ordinary differential equations (ODEs). In section 3, several families of exact solutions are constructed, their asymptotic behavior is investigated and some biological interpretation of the solutions is suggested. We also show that the solutions obtained differ from those found in [8–11] and they cannot be obtained by the standard Lie symmetry method [17–20]. The main results of the paper are summarized in section 4.

2. Reduction of the SKT system (2) to the ODE systems

System (2) can be rewritten in the form

$$\begin{aligned} u_t &= d_1u_{xx} + 2d_{11}uu_{xx} + d_{12}vu_{xx} + d_{12}uv_{xx} + 2d_{11}{u_x}^2 + 2d_{12}u_xv_x + a_1u - b_1u^2 - c_1uv, \\ v_t &= d_2v_{xx} + 2d_{22}vv_{xx} + d_{21}uv_{xx} + d_{21}vu_{xx} + 2d_{22}{v_x}^2 + 2d_{21}u_xv_x + a_2v - c_2v^2 - b_2uv. \end{aligned} \tag{3}$$

Now one sees that it contains only quadratic nonlinearities and has the similar structure to that, which has been analyzed in [14]. Hence we can apply the method of additional generating conditions [14, 15] and use the following conditions:

$$\begin{aligned} \beta_1(t)\frac{du}{dx} + \beta_2(t)\frac{d^2u}{dx^2} + \frac{d^3u}{dx^3} &= 0, \\ \beta_1(t)\frac{dv}{dx} + \beta_2(t)\frac{d^2v}{dx^2} + \frac{d^3v}{dx^3} &= 0, \end{aligned} \tag{4}$$

where $\beta_1(t)$, $\beta_2(t)$ are arbitrary smooth functions at the moment and the variable t is considered as a parameter. Note one can consider this method as an efficient realization of the more general approach described in the book [16]. However, it should be stressed that the method of differential constraints [16] does not suggest any algorithm for how to find appropriate constraints. Our experience says that the linear ordinary differential equations are the most relevant constraints in the case of systems of partial differential equations with quadratic nonlinearities [12, 14, 21].

Depending on $\beta_1(t)$ and $\beta_2(t)$ the general solution of a linear ODE system (4) can have the forms

$$\begin{aligned} u &= \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2, \\ v &= \psi_0(t) + \psi_1(t)x + \psi_2(t)x^2, \end{aligned} \tag{5}$$

if $\beta_1 = \beta_2 = 0$;

$$\begin{aligned} u &= \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)\mathrm{e}^{\gamma(t)x}, \\ v &= \psi_0(t) + \psi_1(t)x + \psi_2(t)\mathrm{e}^{\gamma(t)x}, \end{aligned} \tag{6}$$

if $\beta_1 = 0$;

$$\begin{aligned} u &= \varphi_0(t) + \varphi_1(t)\mathrm{e}^{\gamma_1(t)x} + \varphi_2(t)\mathrm{e}^{\gamma_2(t)x}, \\ v &= \psi_0(t) + \psi_1(t)\mathrm{e}^{\gamma_1(t)x} + \psi_2(t)\mathrm{e}^{\gamma_2(t)x}, \end{aligned} \tag{7}$$

if $D = \beta_2^2 - 4\beta_1 > 0$ and $\gamma_{1,2}(t) = \frac{1}{2}(\pm\sqrt{D} - \beta_2)$, $\gamma_1 \neq \gamma_2$;

$$\begin{aligned} u &= \varphi_0(t) + \mathrm{e}^{\rho x}(\varphi_1(t)\cos\gamma x + \varphi_2(t)\sin\gamma x), \\ v &= \psi_0(t) + \mathrm{e}^{\rho x}(\psi_1(t)\cos\gamma x + \psi_2(t)\sin\gamma x), \end{aligned} \tag{8}$$

if $D < 0$ and $\rho = -\frac{\beta_2}{2}$, $\gamma = \frac{\sqrt{-D}}{2}$;

$$\begin{aligned} u &= \varphi_0(t) + \varphi_1(t)\mathrm{e}^{\gamma(t)x} + x\varphi_2(t)\mathrm{e}^{\gamma(t)x}, \\ v &= \psi_0(t) + \psi_1(t)\mathrm{e}^{\gamma(t)x} + x\psi_2(t)\mathrm{e}^{\gamma(t)x}, \end{aligned} \tag{9}$$

if $D = 0$ and $\gamma_1 = \gamma_2 = \gamma$.

Let us consider (5)–(9) as ansätze containing unknown functions $\varphi_i(t)$ and $\psi_i(t)$, $i = 0, 1, 2$ and find constraints on the coefficients when these ansätze reduce the SKT system (3) to the systems of ODEs with respect to $\varphi_i(t)$ and $\psi_i(t)$. The straightforward calculations, which are rather simple but cumbersome, show that this approach works and we present the results below.

Ansatz (5). The constraints must have the form

$$\begin{aligned} &a_1 = a_2 = a, \quad b_1 = b_2 = b, \quad c_1 = c_2 = c, \\ &d_{11}c - d_{12}b = d_{21}c - d_{22}b, \end{aligned} \tag{10}$$

and

$$\varphi_2 = C\varphi_1, \qquad \psi_1 = -\frac{b}{c}\varphi_1, \qquad \psi_2 = -\frac{b}{c}C\varphi_1, \tag{11}$$

where C is an arbitrary constant. Applying ansatz (5) with the additional constraints (11) to the SKT system (3) with the coefficient restrictions (10), one arrives at the system of ODEs

$$\dot{\varphi}_0 = a\varphi_0 - b\varphi_0^2 - c\varphi_0\psi_0 + 2C\left(2d_{11} - \frac{b}{c}d_{12}\right)\varphi_0\varphi_1 + 2d_{12}C\varphi_1\psi_0 + 2d_1C\varphi_1 + 2\left(d_{11} - \frac{b}{c}d_{12}\right)\varphi_1^2,$$

$$\dot{\psi}_0 = a\psi_0 - c\psi_0^2 - b_2\varphi_0\psi_0 + 2C\left(2\frac{b}{c}d_{22} - d_{21}\right)\varphi_1\psi_0 + 2d_{21}C\frac{b}{c}\varphi_0\varphi_1 + 2d_2C\frac{b}{c}\varphi_1 + 2\left(d_{22}\left(\frac{b}{c}\right)^2 - \frac{b}{c}d_{21}\right)\varphi_1^2,$$

$$\dot{\varphi}_1 = \varphi_1\left(a - b\varphi_0 - c\psi_0 + 12C\left(d_{11} - \frac{b}{c}d_{12}\right)\varphi_1\right). \tag{12}$$

Ansatz (7) with $\gamma_2 = -\gamma_1 = -\gamma \in \mathbf{R}$.

Subcase 1. If the constraints

$$\gamma^2 = \frac{b_1}{4d_{11}} = \frac{c_1}{4d_{12}} = \frac{b_2}{4d_{21}} = \frac{c_2}{4d_{22}} > 0 \tag{13}$$

take place then the system of ODEs

$$\begin{aligned}
\dot{\varphi}_0 &= a_1\varphi_0 - b_1\varphi_0^2 - c_1\varphi_0\psi_0 - 2b_1\varphi_1\varphi_2 - c_1\varphi_2\psi_1 - c_1\varphi_1\psi_2,\\
\dot{\psi}_0 &= a_2\psi_0 - c_2\psi_0^2 - b_2\varphi_0\psi_0 - 2c_2\psi_1\psi_2 - b_2\varphi_1\psi_2 - b_2\varphi_2\psi_1,\\
\dot{\varphi}_1 &= \left(a_1 + \frac{d_1b_1}{4d_{11}}\right)\varphi_1 - \frac{3}{2}b_1\varphi_0\varphi_1 - \frac{3}{4}c_1\varphi_1\psi_0 - \frac{3}{4}c_1\varphi_0\psi_1,\\
\dot{\psi}_1 &= \left(a_2 + \frac{d_2b_1}{4d_{11}}\right)\psi_1 - \frac{3}{2}c_2\psi_0\psi_1 - \frac{3}{4}b_2\varphi_0\psi_1 - \frac{3}{4}b_2\varphi_1\psi_0,\\
\dot{\varphi}_2 &= \left(a_1 + \frac{d_1b_1}{4d_{11}}\right)\varphi_2 - \frac{3}{2}b_1\varphi_0\varphi_2 - \frac{3}{4}c_1\varphi_2\psi_0 - \frac{3}{4}c_1\varphi_0\psi_2,\\
\dot{\psi}_2 &= \left(a_2 + \frac{d_2b_1}{4d_{11}}\right)\psi_2 - \frac{3}{2}c_2\psi_0\psi_2 - \frac{3}{4}b_2\varphi_0\psi_2 - \frac{3}{4}b_2\varphi_2\psi_0
\end{aligned} \tag{14}$$

is obtained.

Subcase 2. The constraints are

$$\begin{gathered}
\gamma^2 = \frac{a_2 - a_1}{d_1 - d_2} > 0, \qquad (d_{22} - d_{12})\kappa = d_{11} - d_{21},\\
\kappa = -\frac{4d_{11}\gamma^2 - b_1}{4d_{12}\gamma^2 - c_1} = -\frac{4d_{21}\gamma^2 - b_2}{4d_{22}\gamma^2 - c_2} = -\frac{d_{11}\gamma^2 - b_1}{d_{22}\gamma^2 - c_2} \neq 0
\end{gathered} \tag{15}$$

and

$$\varphi_2 = C\varphi_1, \qquad \psi_1 = \kappa\varphi_1, \qquad \psi_2 = \kappa C\varphi_1. \tag{16}$$

The relevant system of ODEs reads as

$$\begin{aligned}
\dot{\varphi}_0 &= a_1\varphi_0 - b_1\varphi_0^2 - c_1\varphi_0\psi_0 - 2(b_1 + c_1\kappa)C\varphi_1^2,\\
\dot{\psi}_0 &= a_2\psi_0 - c_2\psi_0^2 - b_2\varphi_0\psi_0 - 2\left(c_2 + \frac{b_2}{\kappa}\right)C\kappa^2\varphi_1^2,\\
\dot{\varphi}_1 &= (a_1 + d_1\gamma^2)\varphi_1 + (2(d_{11}\gamma^2 - b_1) + (d_{12}\gamma^2 - c_1)\kappa)\varphi_0\varphi_1 + (d_{12}\gamma^2 - c_1)\varphi_1\psi_0.
\end{aligned} \tag{17}$$

Ansatz (8) with $\rho = 0$.

Subcase 1. The constraints

$$\gamma^2 = \frac{-b_1}{4d_{11}} = \frac{-c_1}{4d_{12}} = \frac{-b_2}{4d_{21}} = \frac{-c_2}{4d_{22}} > 0 \tag{18}$$

lead to the system of ODEs

$$\begin{aligned}
\dot\varphi_0 &= a_1\varphi_0 - b_1\varphi_0^2 - c_1\varphi_0\psi_0 - \frac{1}{2}b_1\varphi_1^2 - \frac{1}{2}c_1\varphi_1\psi_1 - \frac{1}{2}b_1\varphi_2^2 - \frac{1}{2}c_1\varphi_2\psi_2,\\
\dot\psi_0 &= a_2\psi_0 - c_2\psi_0^2 - b_2\varphi_0\psi_0 - \frac{1}{2}c_2\psi_1^2 - \frac{1}{2}b_2\varphi_1\psi_1 - \frac{1}{2}c_2\psi_2^2 - \frac{1}{2}b_2\varphi_2\psi_2,\\
\dot\varphi_1 &= \left(a_1 + \frac{d_1b_1}{4d_{11}}\right)\varphi_1 - \frac{3}{2}b_1\varphi_0\varphi_1 - \frac{3}{4}c_1\varphi_1\psi_0 - \frac{3}{4}c_1\varphi_0\psi_1,\\
\dot\psi_1 &= \left(a_2 + \frac{d_2b_1}{4d_{11}}\right)\psi_1 - \frac{3}{2}c_2\psi_0\psi_1 - \frac{3}{4}b_2\varphi_0\psi_1 - \frac{3}{4}b_2\varphi_1\psi_0,\\
\dot\varphi_2 &= \left(a_1 + \frac{d_1b_1}{4d_{11}}\right)\varphi_2 - \frac{3}{2}b_1\varphi_0\varphi_2 - \frac{3}{4}c_1\varphi_2\psi_0 - \frac{3}{4}c_1\varphi_0\psi_2,\\
\dot\psi_2 &= \left(a_2 + \frac{d_2b_1}{4d_{11}}\right)\psi_2 - \frac{3}{2}c_2\psi_0\psi_2 - \frac{3}{4}b_2\varphi_0\psi_2 - \frac{3}{4}b_2\varphi_2\psi_0.
\end{aligned} \tag{19}$$

Subcase 2. If constraints (16) and

$$\begin{aligned}
&\gamma^2 = \frac{a_1 - a_2}{d_1 - d_2} > 0, \qquad (d_{22} - d_{12})\kappa = d_{11} - d_{21},\\
&\kappa = -\frac{4d_{11}\gamma^2 + b_1}{4d_{12}\gamma^2 + c_1} = -\frac{4d_{21}\gamma^2 + b_2}{4d_{22}\gamma^2 + c_2} = -\frac{d_{11}\gamma^2 + b_1}{d_{22}\gamma^2 + c_2} \neq 0.
\end{aligned} \tag{20}$$

take place then the system of ODEs

$$\begin{aligned}
\dot\varphi_0 &= a_1\varphi_0 - b_1\varphi_0^2 - c_1\varphi_0\psi_0 + (-2d_{11}\gamma^2 - b_1 + (-2d_{12}\gamma^2 - c_1)\kappa + 2d_{11}\gamma^2C^2\\
&\quad + 2d_{12}\gamma^2C^2\kappa)\varphi_1^2,\\
\dot\psi_0 &= a_2\psi_0 - c_2\psi_0^2 - b_2\varphi_0\psi_0 + ((-2d_{22}\gamma^2 - c_2)\kappa^2 + (-2d_{21}\gamma^2 - b_2)\kappa\\
&\quad + 2d_{21}\gamma^2C^2\kappa + 2d_{22}\gamma^2C^2\kappa^2)\varphi_1^2,\\
\dot\varphi_1 &= (a_1 - d_1\gamma^2)\varphi_1 - (2d_{11}\gamma^2 + 2b_1 + (d_{12}\gamma^2 + c_1)\kappa)\varphi_0\varphi_1 - (d_{12}\gamma^2 + c_1)\varphi_1\psi_0
\end{aligned} \tag{21}$$

is obtained.

Finally, we have established that ansätze (6) and (9) can be applied only in the case $\varphi_2 = \psi_2 = 0$, i.e. if they are reduced to the particular cases of those (5) and (7), respectively.

3. Exact solutions of the SKT system (2) and their properties

To construct exact solutions of the GSKT system (2) one needs to solve the nonlinear systems of ODE derived in section 2. It is well known that nonlinear ODE systems are integrable only in exceptional cases. In this section we demonstrate that exact solutions of system (2) can be constructed in explicit form under further restrictions on its coefficients. It should be stressed that all exact solutions arising in [8–11] have also been constructed under some restrictions on coefficients.

We remind the reader that system (2) similarly to the classical LV system possesses four steady-state solutions

$$\begin{aligned}
&(1)\quad u_0 = v_0 = 0,\\
&(2)\quad u_0 = \frac{a_1}{b_1}, \qquad v_0 = 0,\\
&(3)\quad u_0 = 0 \qquad v_0 = \frac{a_2}{c_2},\\
&(4)\quad u_0 = \frac{a_1c_2 - a_2c_1}{b_1c_2 - b_2c_1}, \qquad v_0 = \frac{a_1b_2 - a_2b_1}{b_2c_1 - b_1c_2},
\end{aligned} \tag{22}$$

which play an important role for the qualitative analysis of the SKT system (2). In fact, it has been proved that $(u, v) \to (u_0, v_0)$ uniformly as $t \to \infty$ if the zero Neumann boundary conditions and the relevant restrictions on the coefficient of (2) take place (see [4] and the references therein). It turns out the exact solutions of (2) in an explicit form can be constructed, which have similar properties. Note we do not apply any boundary conditions to construct these solutions. It should also be stressed that finding spatially non-homogenous steady-state solutions of (2) is a difficult problem because one needs to solve the non-integrable system of two nonlinear second-order ODEs. Nevertheless examples of non-homogenous steady-state solutions will be found below.

Consider system (14) with the coefficients

$$\begin{aligned} & b_1 = b_2 = b, \\ & c_1 = c_2 = c, \\ & a_2 = -\frac{c}{b} a_1, \\ & d_{12} = d_{22} = \frac{c}{b} d_{11}, \\ & d_{21} = d_{11}, \\ & d_2 = d_1 + 4d_{11} \frac{a_1}{b} \left(1 + \frac{c}{b}\right). \end{aligned} \tag{23}$$

If one additionally assumes

$$\begin{aligned} & \psi_0 = \varphi_0 - \frac{a_1}{b}, \\ & \varphi_2 = C\varphi_1, \\ & \psi_1 = -\frac{b}{c}\varphi_1, \\ & \psi_2 = -\frac{b}{c} C\varphi_1, \end{aligned} \tag{24}$$

then this system is reduced to the integrable system of two ODEs. The general solution of this system leads to the exact solution

$$\begin{aligned} u = {} & \frac{1}{K_1 \mathrm{e}^{-\frac{a_1}{b}(b+c)t} + \frac{b}{a_1}} + \exp\left(\left(a_1 + \frac{bd_1}{4d_{11}} + \frac{3}{4}\frac{a_1}{b}c\right)t\right) \\ & \times \left(\mathrm{e}^{\frac{a_1}{b}(b+c)t} + \frac{a_1}{b}K_1\right)^{-\frac{3}{4}} (K_2\exp(\gamma x) + K_3\exp(-\gamma x)), \\ v = {} & \frac{-\left(\frac{a_1}{b}\right)^2 K_1}{\mathrm{e}^{\frac{a_1}{b}(b+c)t} + \frac{a_1}{b}K_1} - \frac{b}{c}\exp\left(\left(a_1 + \frac{bd_1}{4d_{11}} + \frac{3}{4}\frac{a_1}{b}c\right)t\right) \\ & \times \left(\mathrm{e}^{\frac{a_1}{b}(b+c)t} + \frac{a_1}{b}K_1\right)^{-\frac{3}{4}} (K_2\exp(\gamma x) + K_3\exp(-\gamma x)) \end{aligned} \tag{25}$$

of system (2) with restrictions (23). Hereafter K_1, K_2 and K_3 are arbitrary constants.

Asymptotic behavior of solution (25) essentially depends on coefficients (23). There are four cases and two of them are presented below.

Case 1. If

$$\frac{a_1}{b}(b + c) > 0, \qquad a_1 + \frac{bd_1}{d_{11}} < 0, \tag{26}$$

then one obtains

$$(u, v) \to \left(\frac{a_1}{b}, 0\right), \qquad t \to \infty. \tag{27}$$

Using the biological terminology, it means that the competition between the species u and v is rather strong and the species u eventually dominate while the species v die. The final distribution of the species u and v is just case 2 in (22).

Case 2. If

$$\frac{a_1}{b}(b+c) > 0, \qquad d_1 = -\frac{a_1}{b}d_{11}, \tag{28}$$

then

$$(u, v) \to \left(\frac{a_1}{b} + K_2\exp(\gamma x) + K_3\exp(-\gamma x), -\frac{b}{c}K_2\exp(\gamma x) - \frac{b}{c}K_3\exp(-\gamma x)\right),$$
$$t \to \infty. \tag{29}$$

This can be interpreted as an example of the weak competition between the two species. Such competition admits for as long as possible coexistence of the species (the typical real example is parasites and their carriers). It should be stressed that stationary solution (29) is a non-homogenous steady-state solution not belonging to (22).

The next two cases occur under condition $\frac{a_1}{b}(b+c) < 0$ and are quite similar to the cases presented above.

One notes that the ODE system (14) can be essentially simplified by setting $\varphi_2 = \psi_2 = 0$:

$$\begin{aligned}
\dot{\varphi_0} &= a_1\varphi_0 - b_1\varphi_0^2 - c_1\varphi_0\psi_0,\\
\dot{\psi_0} &= a_2\psi_0 - c_2\psi_0^2 - b_2\varphi_0\psi_0,\\
\dot{\varphi_1} &= \left(a_1 + \frac{d_1b_1}{4d_{11}}\right)\varphi_1 - \frac{3}{2}b_1\varphi_0\varphi_1 - \frac{3}{4}c_1\varphi_1\psi_0 - \frac{3}{4}c_1\varphi_0\psi_1,\\
\dot{\psi_1} &= \left(a_2 + \frac{d_2b_1}{4d_{11}}\right)\psi_1 - \frac{3}{2}c_2\psi_0\psi_1 - \frac{3}{4}b_2\varphi_0\psi_1 - \frac{3}{4}b_2\varphi_1\psi_0.
\end{aligned} \tag{30}$$

Obviously the first two equations from (30) can be separately considered as a subsystem. This subsystem possesses four steady-state solutions (u_0, v_0) listed in (22). Substituting any of them into the third and fourth equations of (30), i.e. setting $\varphi_0 = u_0$, $\psi_0 = v_0$, we arrive at the linear ODE system

$$\begin{aligned}
\dot{\varphi_1} &= \left(a_1 + \frac{d_1b_1}{4d_{11}} - \frac{3}{2}b_1u_0 - \frac{3}{4}c_1v_0\right)\varphi_1 - \frac{3}{4}c_1u_0\psi_1,\\
\dot{\psi_1} &= -\frac{3}{4}b_2v_0\varphi_1 + \left(a_2 + \frac{d_2b_1}{4d_{11}} - \frac{3}{2}c_2v_0 - \frac{3}{4}b_2u_0\right)\psi_1.
\end{aligned} \tag{31}$$

The form of the general solution of (31) depends essentially on the values of (u_0, v_0) listed in (22). The case $(u_0, v_0) = (0, 0)$ is rather trivial because it leads to the solution

$$\begin{aligned}
u &= K_1\exp\left(\left(a_1 + \frac{d_1b_1}{4d_{11}}\right)t + \gamma x\right),\\
v &= K_2\exp\left(\left(a_2 + \frac{d_2b_1}{4d_{11}}\right)t + \gamma x\right), \qquad \gamma^2 = \frac{b_1}{4d_{11}} > 0,
\end{aligned} \tag{32}$$

which has very simple behavior. It should be noted that (32) is not a plane wave solution in the usual sense because the components u and v move with the different speeds provided $a_1 + \frac{d_1b_1}{4d_{11}} \neq a_2 + \frac{d_2b_1}{4d_{11}}$.

A more interesting solution occurs if one uses (u_0, v_0) from the second case of (22). Solving (31) with $\left(2 - 3\frac{b_2}{b_1}\right)a_1 + 4a_2 + \frac{(d_2-d_1)b_1}{d_{11}} \neq 0$ and using ansatz (7) with $\varphi_2 = \psi_2 = 0$, we obtain the solution

$$u = \frac{a_1}{b_1} + \left(K_1 \exp\left(\left(a_2 - \frac{3}{4}\frac{b_2}{b_1}a_1 + \frac{d_2 b_1}{4d_{11}}\right)t\right) + K_2 \exp\left(\left(-\frac{1}{2}a_1 + \frac{d_1 b_1}{4d_{11}}\right)t\right)\right)\exp(\gamma x),$$
$$v = K_3 \exp\left(\left(a_2 - \frac{3}{4}\frac{b_2}{b_1}a_1 + \frac{d_2 b_1}{4d_{11}}\right)t + \gamma x\right) \quad (33)$$

(here $K_3 = -K_1 \frac{b_1}{3a_1 c_1}\left(\left(2 - 3\frac{b_2}{b_1}\right)a_1 + 4a_2 + \frac{(d_2-d_1)b_1}{d_{11}}\right)$ and $\gamma^2 = \frac{b_1}{4d_{11}} > 0$) of the SKT system

$$\begin{aligned} u_t &= \left[\left(d_1 + d_{11}u + d_{11}\frac{c_1}{b_1}v\right)u\right]_{xx} + u(a_1 - b_1 u - c_1 v), \\ v_t &= \left[\left(d_2 + d_{11}\frac{b_2}{b_1}u + d_{11}\frac{c_2}{b_1}v\right)v\right]_{xx} + v(a_2 - b_2 u - c_2 v). \end{aligned} \quad (34)$$

The general solution (31) with $\left(2 - 3\frac{b_2}{b_1}\right)a_1 + 4a_2 + \frac{(d_2-d_1)b_1}{d_{11}} = 0$ leads to the following exact solution of the SKT system (34):

$$\begin{aligned} u &= \frac{a_1}{b_1} + (K_2 - K_1 t)\exp\left(\left(-\frac{1}{2}a_1 + \frac{d_1 b_1}{4d_{11}}\right)t + \gamma x\right), \\ v &= K_1 \frac{4}{3}\frac{b_1}{a_1 c_1}\exp\left(\left(-\frac{1}{2}a_1 + \frac{d_1 b_1}{4d_{11}}\right)t + \gamma x\right). \end{aligned} \quad (35)$$

One notes asymptotic behavior of solutions (33) and (35) as follows:

$$(u, v) \to \left(\frac{a_1}{b_1}, 0\right), \qquad t \to \infty, \quad (36)$$

if the restrictions $4a_2 + \frac{d_2 b_1}{d_{11}} < 3\frac{b_2}{b_1}a_1$, $\frac{d_1 b_1}{d_{11}} < 2a_1$ and $\frac{d_1 b_1}{d_{11}} < 2a_1$ take place, respectively. So, one sees that these solutions describe the strong competition between the species u and v.

In quite a similar way one constructs exact solutions of (2) with the coefficient constraints (13) using the values of (u_0, v_0) listed in cases 3 and 4 of (22).

The ODE system (17) can also be solved if one sets the additional restrictions on the coefficients. In the particular case, setting

$$b_1 = b_2 = b, \qquad c_1 = c_2 = c, \qquad d_{12} = d_{22} = \frac{c}{b}d_{11}, \qquad d_{21} = d_{11}, \quad (37)$$

the reduction constraints (15) are satisfied with $\kappa = -\frac{b}{c}$ and (17) takes the form

$$\begin{aligned} \dot{\varphi}_0 &= a_1\varphi_0 - b\varphi_0^2 - c\varphi_0\psi_0, \\ \dot{\psi}_0 &= a_2\psi_0 - c\psi_0^2 - b\varphi_0\psi_0, \\ \dot{\varphi}_1 &= (a_1 + d_1\gamma^2)\varphi_1 + (d_{11}\gamma^2 - b)\varphi_0\varphi_1 + \frac{c}{b}(d_{11}\gamma^2 - b)\varphi_1\psi_0. \end{aligned} \quad (38)$$

Now a particular solution of (38) can be easily constructed and we arrive at the exact solution

$$\begin{aligned} u = {} & \frac{K_1}{e^{(a_2-a_1)t} + \frac{b}{a_1}K_1} + \exp\left(\left(d_1\gamma^2 + \frac{a_1}{b}d_{11}\gamma^2\right)t\right) \\ & \times \left(e^{(a_2-a_1)t} + \frac{b}{a_1}K_1\right)^{\frac{1}{b}d_{11}\gamma^2 - 1}(K_2\exp(\gamma x) + K_3\exp(-\gamma x)), \\ v = {} & \frac{\frac{a_2}{c}e^{(a_2-a_1)t}}{e^{(a_2-a_1)t} + \frac{b}{a_1}K_1} - \frac{b}{c}\exp\left(\left(d_1\gamma^2 + \frac{a_1}{b}d_{11}\gamma^2\right)t\right) \\ & \times \left(e^{(a_2-a_1)t} + \frac{b}{a_1}K_1\right)^{\frac{1}{b}d_{11}\gamma^2 - 1}(K_2\exp(\gamma x) + K_3\exp(-\gamma x)), \end{aligned} \quad (39)$$

of the SKT system

$$\begin{aligned} u_t &= \left[\left(d_1 + d_{11}u + d_{11}\frac{c}{b}v\right)u\right]_{xx} + u(a_1 - bu - cv), \\ v_t &= \left[\left(d_2 + d_{11}u + d_{11}\frac{c}{b}v\right)v\right]_{xx} + v(a_2 - bu - cv), \end{aligned} \tag{40}$$

where $\frac{a_2-a_1}{d_1-d_2} \equiv \gamma^2 > 0$.

Remark 1. Solution (39) with $a_2 > a_1$ and $\frac{b}{a_1}K_1 < 0$ blows up for the finite time $t_{\max} = (a_2 - a_1)^{-1}\ln|\frac{b}{a_1}K_1|$. The exact solutions presented below possess the same property if the appropriate coefficient restrictions take place.

Asymptotic behavior of solution (39) again depends on the coefficients and there are four cases. However only two of them are essentially different and they are presented below.

Case A. If

$$a_2 < a_1, \qquad d_1 + \frac{a_1}{b}d_{11} < 0, \tag{41}$$

then

$$(u, v) \to \left(\frac{a_1}{b}, 0\right), \qquad t \to \infty. \tag{42}$$

Case B. If

$$a_2 < a_1, \qquad d_1 = -\frac{a_1}{b}d_{11}, \tag{43}$$

then

$$(u, v) \to \left(\frac{a_1}{b} + K_4\exp(\gamma x) + K_5\exp(-\gamma x), -\frac{b}{c}K_4\exp(\gamma x) - \frac{b}{c}K_5\exp(-\gamma x)\right), \quad t \to \infty, \tag{44}$$

where $K_4 = K_2\left(\frac{a_1}{bK_1}\right)^{\frac{a_2d_{11}+bd_2}{a_1d_{11}+bd_2}}$, $K_5 = K_3\left(\frac{a_1}{bK_1}\right)^{\frac{a_2d_{11}+bd_2}{a_1d_{11}+bd_2}}$ are arbitrary constants because there are no restrictions on the constants K_2 and K_3. Obviously one can provide the same biological interpretation of (42) and (44) as was done above. An example of solution (39), describing the competition between the two species when the species u eventually dominate while the species v die, is presented in figure 1. Another solution, possessing asymptotic behavior (44), is shown in figure 2. This solution approximately describes coexistence of the species by the spatial segregation of habitat. One sees the steady-state segregation of densities of two competing species: the species u have favorableness of habitat at a position x situated in the left-hand side of the interval [0, 0.8], while the species v have one situated in the right-hand side of this interval.

Consider ODE system (19). It turns out that the coefficient restrictions (23) and assumptions (24) reduce (19) to an integrable system containing only two ODEs. Simultaneously constraints (18) are reduced to the form $\gamma^2 = -\frac{b}{4d_{11}} > 0$. Hence the exact solution

$$u = \frac{1}{K_1 e^{-\frac{a_1}{b}(b+c)t} + \frac{b}{a_1}} + \exp\left(\left(a_1 + \frac{bd_1}{4d_{11}} + \frac{3}{4}\frac{a_1}{b}c\right)t\right) \times \left(e^{\frac{a_1}{b}(b+c)t} + \frac{a_1}{b}K_1\right)^{-\frac{3}{4}}(K_2\cos(\gamma x) + K_3\sin(\gamma x)),$$

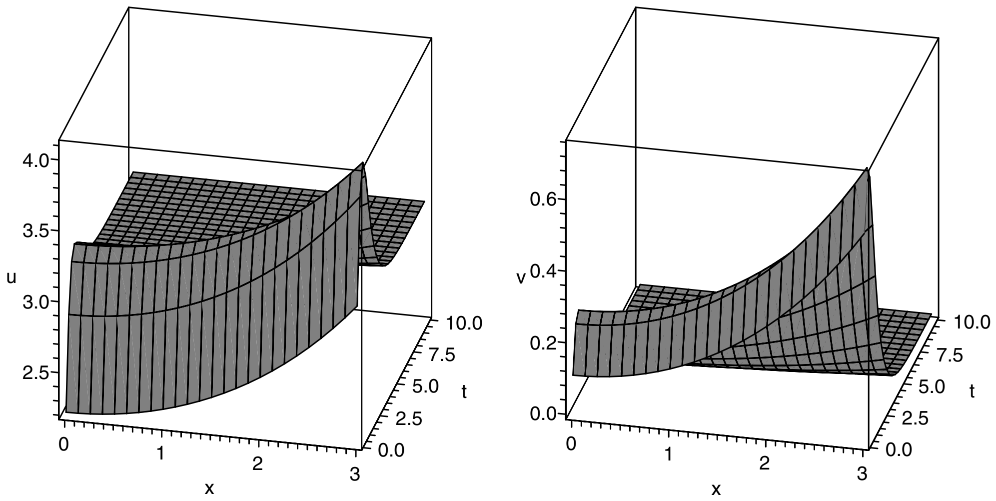

Figure 1. Exact solution (39) with $a_1 = 3, a_2 = 1, b = 1, c = -2, d_1 = 2, d_2 = 10, d_{11} = -2, K_1 = 3, K_2 = 1, K_3 = 1$.

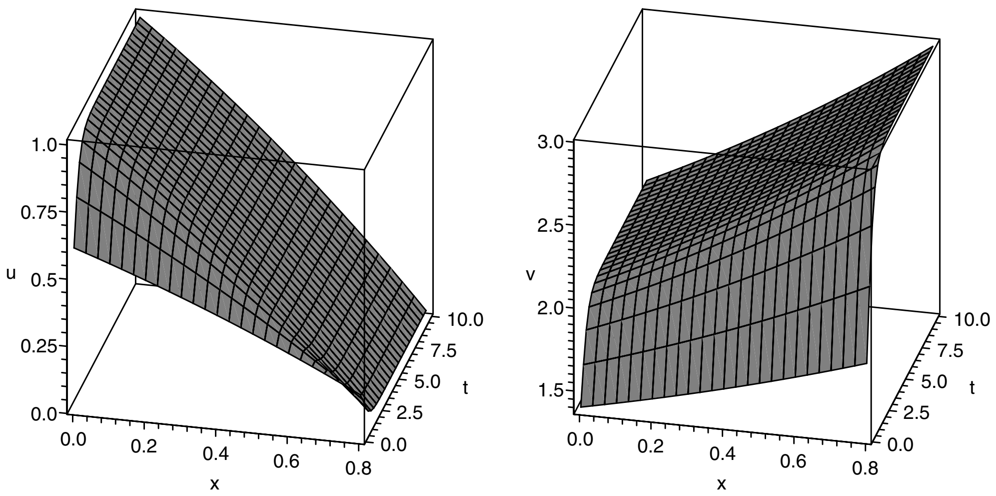

Figure 2. Exact solution (39) with $a_1 = 3, a_2 = 1, b = 1, c = 1, d_1 = 2, d_2 = 10, d_{11} = -\frac{2}{3}, K_1 = 3, K_2 = 0, K_3 = -2$.

$$v = \frac{-\left(\frac{a_1}{b}\right)^2 K_1}{\mathrm{e}^{\frac{a_1}{b}(b+c)t} + \frac{a_1}{b} K_1} - \frac{b}{c} \exp\left(\left(a_1 + \frac{bd_1}{4d_{11}} + \frac{3}{4}\frac{a_1}{b}c\right)t\right)$$
$$\times \left(\mathrm{e}^{\frac{a_1}{b}(b+c)t} + \frac{a_1}{b} K_1\right)^{-\frac{3}{4}} (K_2 \cos(\gamma x) + K_3 \sin(\gamma x)) \qquad (45)$$

of the SKT system (40) with $a_2 = -\frac{c}{b} a_1, d_2 = d_1 + 4d_{11}\frac{a_1}{b}\left(1 + \frac{c}{b}\right)$ and $\frac{b}{4d_{11}} = -\gamma^2 < 0$ has been found.

In contrast to the solutions listed above, this is periodic with respect to the variable x. Solution (45) under the coefficient restrictions (26) possesses the asymptotic behavior (27).

The coefficient restrictions (28) lead to the periodic asymptote

$$(u, v) \to \left(\frac{a_1}{b} + K_2\cos(\gamma x) + K_3\sin(\gamma x), -\frac{b}{c}K_2\cos(\gamma x) - \frac{b}{c}K_3\sin(\gamma x)\right), \qquad t \to \infty. \tag{46}$$

This can be interpreted that the densities of two competing species tend (with time) to a periodical distribution in space. The competition is rather weak and leads to the coexistence of two species.

Finally, let us consider the ODE system (21) with $C = 0$. To satisfy the restrictions on coefficients (20) one may again use coefficients (37) and then system (21) takes the form

$$\begin{aligned} \dot\varphi_0 &= a_1\varphi_0 - b\varphi_0^2 - c\varphi_0\psi_0, \\ \dot\psi_0 &= a_2\psi_0 - c\psi_0^2 - b\varphi_0\psi_0, \\ \dot\varphi_1 &= (a_1 - d_1\gamma^2)\varphi_1 - (d_{11}\gamma^2 + b)\varphi_0\varphi_1 - \frac{c}{b}(d_{11}\gamma^2 + b)\varphi_1\psi_0. \end{aligned} \tag{47}$$

Using the solution of (47), formulae (16) with $C = 0$ and ansatz (8) with $\rho = 0$, $\gamma^2 = \frac{a_1 - a_2}{d_1 - d_2}$, we arrive at the exact solution

$$\begin{aligned} u &= \frac{K_1}{e^{(a_2-a_1)t} + \frac{b}{a_1}K_1} + \exp\left(-\left(d_1\gamma^2 + \frac{a_1}{b}d_{11}\gamma^2\right)t\right) \\ &\quad\times\left(e^{(a_2-a_1)t} + \frac{b}{a_1}K_1\right)^{-\frac{1}{b}d_{11}\gamma^2 - 1} K_2\cos(\gamma x), \\ v &= \frac{\frac{a_2}{c}e^{(a_2-a_1)t}}{e^{(a_2-a_1)t} + \frac{b}{a_1}K_1} - \frac{b}{c}\exp\left(-\left(d_1\gamma^2 + \frac{a_1}{b}d_{11}\gamma^2\right)t\right) \\ &\quad\times\left(e^{(a_2-a_1)t} + \frac{b}{a_1}K_1\right)^{-\frac{1}{b}d_{11}\gamma^2 - 1} K_2\cos(\gamma x) \end{aligned} \tag{48}$$

of the SKT system (40) with $\frac{a_1 - a_2}{d_1 - d_2} > 0$.

In the case of constraints

$$a_2 < a_1, \qquad d_1 + \frac{a_1}{b}d_{11} > 0, \tag{49}$$

solution (48) possesses the asymptotic behavior (42). An example of solution (48) describing the competition between the two species when the species u eventually dominate while the species v die, is presented in figure 3.

In the case of constraints (43), solution (48) possesses the asymptotic behavior

$$(u, v) \to \left(\frac{a_1}{b} + K_3\cos(\gamma x), -\frac{b}{c}K_3\cos(\gamma x)\right), \qquad t \to \infty, \tag{50}$$

where $K_3 = K_2\left(\frac{a_1}{bK_1}\right)^{\frac{a_2 d_{11} + b d_2}{a_1 d_{11} + b d_2}}$. So we again obtain a periodical distribution with respect to the variable x like for solution (45) under restrictions (28). One notes that solution (48) under restrictions (43) approximately describes coexistence of two species by the spatial segregation of habitat. An example of solution (48) with such behavior is presented in figure 4.

In conclusion of this section we want to show that the exact solutions obtained above cannot be found by the methods used in the recent papers [8–11]. The authors of [8, 11] constructed the solutions starting from the well-known plane wave ansatz

$$u = \varphi(x - kt), \qquad v = \psi(x - kt), \tag{51}$$

where the constant k means the wave speed. Obviously, the solutions obtained above possess a more complicated structure. In [9] an additional relation between the functions u and v

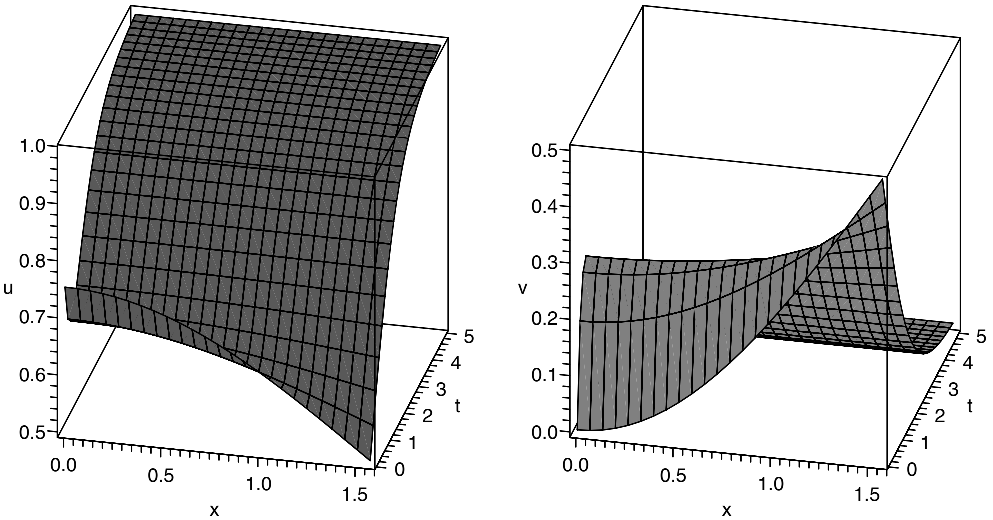

Figure 3. Exact solution (48) with $a_1 = 2, a_2 = 1, b = 2, c = 1, d_1 = 2, d_2 = 1, d_{11} = 2, K_1 = 1, K_2 = 1$.

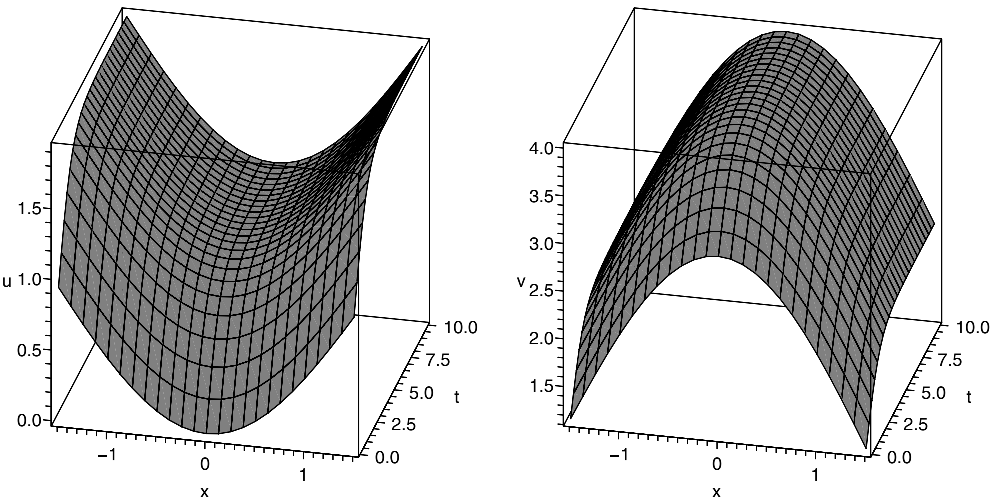

Figure 4. Exact solution (48) with $a_1 = 2, a_2 = 1, b = 1, c = 0.5, d_1 = 2, d_2 = 1, d_{11} = -1, K_1 = 2, K_2 = -1$.

(see formula (49) therein) was used to construct exact solutions of the SKT system (2) with $d_{11} = d_{22} = 0$. One notes that the solutions found here do not satisfy relation (49) [9].

In [10] the classical Lie symmetry method [17–20] was used to construct exact solutions of the diffusive LV system. It turns out that application of this approach to the SKT system (2) is a rather non-trivial and cumbersome task and will be presented in a forthcoming paper. However, we want to show now that the solutions obtained here cannot be found using Lie symmetries. In fact, we were able to construct exact solutions only under the coefficient restrictions presented above. One notes that formulae (25), (39), (45) and (48) present exact solutions for the SKT system (40), which is a subcase of (2) and contains only seven arbitrary

Table 1. Lie symmetries of system (40).

No.	Systems	Restrictions	Basic operators of MAI
1.	$u_t = [(d_{11}u + \frac{c}{b}d_{11}v)u]_{xx} + u(a - bu - cv)$ $v_t = [(d_{11}u + \frac{c}{b}d_{11}v)v]_{xx} + v(2a - bu - cv)$	$a \neq 0$	$P_t = \partial_t,\ P_x = \partial_x,$ $Q_1 = \mathrm{e}^{-at}$ $\times(\partial_t + a(u - \frac{c}{b}v)\partial_u + 2av\partial_v)$
2.	$u_t = [(d + \frac{b}{a}\mathrm{d}u + \frac{c}{a}\mathrm{d}v)u]_{xx} + u(a - bu - cv)$ $v_t = [(2d + \frac{b}{a}\mathrm{d}u + \frac{c}{a}\mathrm{d}v)v]_{xx} - v(a + bu + cv)$	$ad \neq 0$	$P_t,\ P_x,$ $Q_2 = \mathrm{e}^{-at}$ $\times(\partial_t + 2au\partial_u + a(\frac{a}{c} - \frac{b}{c}u + v)\partial_v)$
3.	$u_t = [(d + d_{11}u + \frac{c}{b}d_{11}v)u]_{xx} + u(a - bu - cv)$ $v_t = [(d + d_{11}u + \frac{c}{b}d_{11}v)v]_{xx} + v(a - bu - cv)$	$ad \neq 0$	$P_t,\ P_x,$ $J_u = u(c\partial_u - b\partial_v),$ $J_v = v(c\partial_u - b\partial_v)$
4.	$u_t = [(d_{11}u + \frac{c}{b}d_{11}v)u]_{xx} + u(a - bu - cv)$ $v_t = [(d_{11}u + \frac{c}{b}d_{11}v)v]_{xx} + v(a - bu - cv)$	$a \neq 0$	$P_t,\ P_x,\ J_u,\ J_v,$ $Q_3 = \mathrm{e}^{-at}(\partial_t + a(u\partial_u + v\partial_v))$
5.	$u_t = [(d_{11}u + \frac{c}{b}d_{11}v)u]_{xx} - u(bu + cv)$ $v_t = [(d_{11}u + \frac{c}{b}d_{11}v)v]_{xx} - v(bu + cv)$		$P_t,\ P_x,\ J_u,\ J_v,$ $D = t\partial_t - u\partial_u - v\partial_v$

parameters. Now let us formulate a theorem which gives complete information on the Lie symmetry of (40).

Theorem 1. *All possible maximal algebras of invariance (MAI) of system (40) with* $d_{11} \neq 0$ *and* $a^2 + b^2 + c^2 \neq 0$ *are presented in table 1. Any other system of the form (40) either is reduced by the renaming*

$$\begin{aligned} u &\to v \\ v &\to u \end{aligned} \tag{52}$$

to one of those given in table 1 or is invariant under the two-dimensional Lie algebra with the basic operators $P_t = \partial_t,\ P_x = \partial_x$.

Proof. It is based on the standard Lie scheme [17–20] and omitted here because of its bulk. □

Remark 2. In the case $d_{11} = 0$ system (40) is a particular case of the diffusive LV system and its Lie symmetries were described in [10] and in the case $a^2 + b^2 + c^2 = 0$ this system is no longer a reaction-diffusion system but a cross-diffusion system without any reaction terms.

Remark 3. In the cases 3–5 of table 1 the corresponding systems with cross-diffusion can be reduced to semi-coupled those of the form

$$\begin{aligned} w_t &= \left[\left(d + \frac{d_{11}}{b}w\right)w\right]_{xx} + w(a - w), \\ u_t &= \left[\left(d + \frac{d_{11}}{b}w\right)u\right]_{xx} + u(a - w), \end{aligned}$$

where $w = bu + cv$. In this system the first equation is the well-known Fisher equation with nonconstant diffusivity and its exact solutions were found in [22] (in the case $d = 0$) and [9] (in the case $d \neq 0$).

Taking into account this theorem one can claim that there are only five cases listed in table 1 when the Lie method allows one to construct ansätze, which differ from the plane wave ansatz (51). It turns out that any system arising in these cases is not relevant to those which we were able to construct exact solutions for. Thus, these solutions cannot be obtained using the operators arising in table 1.

Let us take, for example, solution (25). One easily notes that this solution cannot be obtained by using the plane wave ansatz (51) provided $K_1^1 + K_2^2 \neq 0$ or $K_1^1 + K_3^2 \neq 0$. On the other hand, the SKT system (2) with restrictions (23) does not belong to any system from table 1 (of course, for some *correctly-specified* values of the parameters a, b, c, d_1 and d_{11} it can be done but not for *arbitrary* values). Thus, system (2) with restrictions (23) is invariant under the two-dimensional Lie algebra with the basic operators $P_t = \partial_t, P_x = \partial_x$, therefore ansatz (51) can only be derived. It means that solution (25) is not obtainable by the well-known Lie machinery. In the quite similar way the same conclusion can be derived for solutions (39), (45) and (48).

Finally, exact solutions (32), (33) and (35) are also not obtainable by the Lie symmetry method because the SKT system (34) admits only the two-dimensional Lie algebra $\langle P_t, P_x\rangle$ provided its coefficients are arbitrary constants. Of course, these solutions with correctly-specified coefficients (for example, solution (32) for (34) with $a_1 + \frac{d_1 b_1}{4d_{11}} = a_2 + \frac{d_2 b_1}{4d_{11}}$) can also be reduced to the form of plane wave solutions (51) but not in the general case.

4. Conclusions

In this paper the method of additional generating conditions [14, 15] was successfully applied to find exact solutions in the explicit form for the SKT system (2), which was extensively studied during last few decades. We have established several ansätze reducing systems (2) to the relevant systems of ODEs if the coefficient of (2) satisfies the appropriate conditions. Nevertheless the ODE systems obtained are essentially nonlinear, they were solved under further restrictions on the coefficients arising in (2). Hence several families of exact solutions of the SKT system (2) were constructed and they are given by the formulae (25), (32), (33), (35), (39), (45) and (48).

The asymptotic behavior of the solutions obtained was further investigated. We have established the conditions when these solutions tend (if time $t \to \infty$) to steady-state points of (2). Moreover, new nonuniform stationary distributions were found (see formulae (29), (44) and (46)), which are asymptotical limits of solutions (25), (39) and (45), if the relevant coefficient restrictions are valid. From the theoretical point of view it means that the solutions obtained (with the relevant coefficient restrictions) are rather stable with respect to sufficiently small perturbations of the initial profiles generated by them and therefore any perturbed solution (one can be obtained by numerical simulations) should possess similar asymptotic properties to the relevant exact solution. From the applicability point of view it means that such solutions describe typical forms of the competition between the species u and v, which are predicted by their biological nature. In the particular case, the non-homogenous (nonuniform) solutions obtained may approximately describe coexistence of two species by the spatial segregation of habitat (see figures 2 and 4). Of course, the problem how to satisfy the relevant boundary conditions (typically they are the zero Neumann conditions) is another challenge and we do not discuss this here.

The work is still in progress. In the particular case, we are going to finish a complete description of Lie symmetries of the SKT system (2), which is a non-trivial problem since the system contains 12 arbitrary parameters, and to apply the symmetries obtained for finding new solutions of this system.

Acknowledgments

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